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Fundamental Solution for a Nonlocal Second Order Problem of the Hyperbolic Type

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Abstract. This text discusses the importance and challenges of studying nonlocal boundary value problems, which arise frequently in applied fields such as soil moisture transfer, thermophysics, and diffusion processes. The paper emphasizes the complexity of these problems due to their nonlocal nature, which differs from traditional local boundary value problems. Despite the abundance of research in this area, particularly the works of A.M. Nakhushev and others, mathematical tools to handle nonlocal problems are insufficient, especially when addressing concepts like conjugacy or duality. These issues are particularly significant when coefficients are nonsmooth or measurable. The work introduces the concept of a conjugate problem and the fundamental solution, which generalizes existing functions like Green's and Riemann's functions. The study of these solutions for nonlocal boundary value problems is essential for advancing both theoretical understanding and practical applications.

Keywords: Boundary value, Green function, Hyperbolic type, Nonlocal, Riemann function.

1. INTRODUCTION

The appearnce of such features as loading, integral expression, etc., which are the cause of the nonlocality of boundary value problems, is quite often expected in applications. The noted features cannot be ignored or excluded, and therefore, such nonlocal boundary value problems should be investigated in the form in which they occur.

Over the past few decades, research on nonlocal boundary value problems has become quite common in the scientific literature. The reason for this is the large number of works of an applied nature by different autors. Of these, we can note, for example, the works of A.M.Nakhushev related to the processes of moisture transfer in the soil [1 − 5]. Further, it should be noted the works of A.F.Chudnovsky devoted to the problem of soil moisture and soil thermophisics [6,7]. In addition, the works of A.V.Pshu [8,9] and A.A.Andreyev [10,11] concerning diffusion processes deserve attention and are of scientific interest to researchers involved in the processes of flow of liquids and gases.

The processes of such content are relevant and subject to detailed study. However, the nonlocality of the mathematical discription of applied problems makes it different for such problems some important issues that have been sufficiently studied for local boundary value problems. Namely, conjugacy or duality has been well studied for local linear boundary value problems, both for the ordinary differential equations, and for the partial differential equations.

The study of this issue for linear nonlocal problems encounters serious difficulties of a constructive nature. More precisely, traditional methods of studying linear local problems do not allow us to study linear nonlocal problems using a similar calculus scheme. The lack and inferiority of mathematical operations and tools makes it impossible to introduce the concept of a conjugate problem and to use it for linear nonlocal boundary value problems with both concentrated and distributed parameters.

It should be especially marked the works of A.M.Nakhushev devoted to processes of soil moisture transfer [1 − 5], related with Aller's equation. In this works for similar boundary value problems there is studied the issues of the approach to a solution and correctness of a problem solving. In this works thow nonlocal boundary value problems are linear, but absence of subject mathematical tools does not allow to study by the traditional method the duality problem for the nonlocal boundary value problems and even in the case of locality, if the coefficients are nonsmooth, and also are meassurable.

The concept of a conjugate problem or equation is an important step in the study of different linear boundary value problems. Related to this concept is the concept of a fundamental solution for a linear boundary value problem, as well as the possibility of an integral representation for the solution of this boundary value problem.

Integral representation of solution of a linear boundary value problem related with the concept of the fundamental solution which are introduced for this boundary value problem. Fundamental solutions of linear boundary value problems for linear differential operator equations with constant and sufficiently smooth coefficients there are studied by the classic methods and by generalized functions methods in the works [12 − 16] and etc. The concept of the fundamental solution first there has been introduced in works [17,18] for some linear operator equations, given in a Banach space of smooth functions. Some important applications of this concept given in the works [19 − 24] for local and nonlocal hyperbolic, pseudoparabolic and functional differential problems with generally nonsmooth coefficients.

In the suggested works there are used some concepts and taken structural reconisscence of the considering linear nonlocal problem into account $[17-24]$. Using here of discovered isomorphism between spaces allowed also to discover the structural according between Sobolev space and conjugate space, which is important for study of the duality for linear local problems.

In this work there is noted the linear operator which implements isomorphism between responding spaces. Using this isomorphism, it has become possible to introduce the concept of a conjugate problem for the considered linear nonlocal problem with multipoint boundary conditions. After this, there has been introduced the concept of a fundamental solution for the abovementioned nonlocal boundary problem.

Note that a concept of a fundamental solution introducing here generalises the concepts such as the Green's function and the Riemann function. In other words, the concepts of the Green's function and the Riemann function are special cases of the fundamental solution introduced in this work.

2. FORMULATİON OF THE PROBLEM.

In this paper the second order equations system

$$(V_{1,1}z)(t,x) \equiv z_{tx}(t,x) + z(t,x)A_0(t,x) + z_x(t,x)A_1(t,x) +$$
$$+z_t(t,x)A_2(t,x) + \int_T z_\tau(\tau,h(t,x))K(\tau;t,x)d\tau = g_3(t,x),$$
$$(t,x) \in D = T \times X;\ \ T=[t_0,t_1],\ X=[x_0,x_1], \qquad (1)$$

is considered under nonlocal boundary conditions $[17-24]$

$$(V_{1,0}z)(t) \equiv \sum_{j=1}^{m}\left[z_t(t,\xi_j)\alpha_j(t) + z(t,\xi_j)\beta_j(t)\right] = g_2(t),\ \ t \in T, \qquad (2)$$
$$(V_{0,1}z)(x) \equiv z_x(t_0,x) = g_1(x),\ \ x \in X, \qquad (3)$$
$$V_{0,0}z \equiv z(t_0,x_0) = g_0. \qquad (4)$$

Here: $A_0(t,x), A_1(t,x), A_2(t,x)$ are given $n \times n$ matrices, where $A_0 \in L_{p,n\times n}(D)$, i.e., with elements from $L_p(D)$, $1 \le p \le \infty$; there exist such functions $a_1 \in L_p(T)$, $a_2 \in L_p(X)$ that $\|A_1(t,x)\| \le a_1(t)$, $\|A_2(t,x)\| \le a_2(x)$ almost everywhere on D; $K(\tau;t,x)$ is given $n \times n$ matrix, such that $K(\cdot\,;t,x) \in L_{q,n\times n}(T)$ for almost every $(t,x) \in D$, $q = p/(p-1)$, moreover, the norm $\|K(\cdot\,;t,x)\|_{L_{q,n\times n}(T)}$ as function of $(t,x) \in D$ belongs to space $L_p(D)$; $h(t,x)$ is given measurable function on D, for which $h(t,x) \in X$ for almost every $(t,x) \in D$; $g_3(t,x), g_2(t), g_1(x)$, g_0 are given line $n-$ vectors, such that $g_3 \in L_{p,n}(D)$, $g_2 \in L_{p,n}(T)$, $g_1 \in L_{p,n}(X)$, $g_0 \in R^n$, i.e., with elements from $L_p(D), L_p(T), L_p(X), R$, respectively; $\alpha_j(t), \beta_j(t) -$ are given $n \times n$ matrices and $\alpha_j \in L_{\infty,n\times n}(T)$, $\beta_j \in L_{p,n\times n}(T)$; $\xi_j \in X$, $j = 1,\dots,m$, are given numbers.

The considered linear nonlocal boundary value problem with multipoint boundary conditions is one of various classes of nonlocal boundary value problems. Similar to this also other interesting classes of nonlocal boundary value problems can be found in various applications. For example, in problems of soil moisture and soil thermophisics $[6,7]$, in the study of problems of forcasting soil moisture and grandwater movement $[2-4]$, in works related to diffusion processes $[6,7,8-10]$, in the study of biological and medical processes $[5]$.

Under abovementioned conditions on the data of the problem (1)-(4) we can assume its solution from the Sobolev space $W_{p,n}(D)$ of line $n-$ vector-functions $z \in L_{p,n}(D)$ obtaining general derivatives $z_t, z_x, z_{tx} \in L_{p,n}(D)$ in the Sobolev sense $[13]$. In other words, the operator $V = (V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1})$ of the boundary value problem has (1)-(4) been defined on $W_{p,n}(D)$ and acts into the space $\Delta_{p,n}(D) = R^n \times L_{p,n}(X) \times L_{p,n}(T) \times L_{p,n}(D)$.

3. GENERAL FORM OF THE LINEAR BOUNDARY FUNCTIONALS

For studyof the conjugacu of the problem (1)-(4) we use an isomorphism, which is implemented by the operator $Nz \equiv (z(t_0,x_0), z_x(t_0,x), z_t(t,x_0), z_{tx}(t,x))$ from $W_{p,n}(D)$ onto the space $\Delta_{p,n}(D)$ of fours $\varphi = (\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ $[17-24]$. Take into account for this isomorphism also opposite description

$$z(t,x) = (N^{-1}\varphi)(t,x) \equiv \varphi_0 + \int_{x_0}^{x} \varphi_1(\xi)d\xi + \int_{t_0}^{t} \varphi_2(\tau)d\tau +$$

$$+\int_{t_0}^{t}\int_{x_0}^{x}\varphi_3(\tau,\xi)d\tau d\xi,\quad (t,x)\in D. \qquad (5)$$

Using the operator $V=(V_{0,0},V_{0,1},V_{1,0},V_{1,1})$ of the problem (1)-(4) acting from $W_{p,n}(D)$ into the space $\Delta_{p,n}(D)$ we can write this problem as one operator equation

$$Vz=g,\quad z\in W_{p,n}(D), \qquad (6)$$

where $g=(g_0,g_1,g_2,g_3)$ is given element from $\Delta_{p,n}(D)$.

With help of abovementioned isomorphism we can also write the equation (6) or the problem (1)-(4)) in following equivalent form

$$\vartheta\varphi=g,\quad \varphi\in\Delta_{p,n}(D), \qquad (7)$$

where operator $\vartheta=(\vartheta_{0,0},\vartheta_{0,1},\vartheta_{1,0},\vartheta_{1,1})$ is defined like $\vartheta=VN^{-1}$.

Note that every linear bounded functional f defined on $\Delta_{p,n}(D)$ for $1\le p<\infty$ can be represented in the form

$$f(\varphi)=\varphi_0\cdot f_0'+\int_X\varphi_1(\xi)\cdot f_1'(\xi)d\xi+\int_T\varphi_2(\tau)\cdot f_2'(\tau)d\tau+$$

$$+\iint_D\varphi_3(\tau,\xi)f_3'(\tau,\xi)d\tau d\xi, \qquad (8)$$

where $f=(f_0,f_1,f_2,f_3)$ is some element of the space $\Delta_{q,n}(D)=R^n\times L_{q,n}(X)\times L_{q,n}(T)\times L_{q,n}(D)$ with adjoint number $q=p/(p-1)$ for $1<p<\infty$ and $q=\infty$ for $p=1$, $(\cdot)'$ is transponeering. It seems that for $1\le p<\infty$ the space $\Delta_{q,n}(D)$ is adjoint space to the $\Delta_{p,n}(D)$. Although in case $p=\infty$ the conjugate space $(\Delta_{\infty,n}(D))^*$ is a wider space of measures than $\Delta_{1,n}(D)$, but we assume in this case as well as $\Delta_{1,n}(D)$ like conjugate space.

Taking the isomorphism implementing by the operator $Nz\equiv(z(t_0,x_0),z_x(t_0,x),z_t(t,x_0),z_{tx}(t,x))$ into account we assume (8) as well as linear bounded functional defined on $W_{p,n}(D)$ and the space $\Delta_{q,n}(D)$ (with $q=p/(p-1)$ for every $1\le p\le\infty$) as adjoint space to $W_{p,n}(D)$: $\Delta_{q,n}(D)=W_{p,n}^*(D)$.

4. ADJOINT PROBLEM AND ADJOINT OPERATOR

Now consider any functional $f\in\Delta_{q,n}(D)$ on values of the operator Vz:

$$f(Vz)=(V_{0,0}z)f_0'+\int_X(V_{0,1}z)(\xi)f_1'(\xi)d\xi+$$

$$+\int_T(V_{1,0}z)(\tau)f_2'(\tau)d\tau+\iint_D(V_{1,1}z)(\tau,\xi)f_3'(\tau,\xi)d\tau d\xi. \qquad (9)$$

Due to abovementioned isomorphism and linearity of the operators $V_{0,0}(z),V_{0,1}(z),V_{1,0}(z),V_{1,1}(z)$, the expression (9) may be considered as well as linear boundary functional defined on the fours $\varphi=(\varphi_0,\varphi_1(x),\varphi_2(t),\varphi_3(t,x))$ of the space $\Delta_{p,n}(D)=R^n\times L_{p,n}(X)\times L_{p,n}(T)\times L_{p,n}(D)$. In other words, we can transform the last expression (9) to the following view:

$$f(Vz)=\varphi_0W_{0,0}f'+\int_X\varphi_1(\xi)(W_{0,1}f')(\xi)d\xi+$$

$$+\int_T \varphi_2(\tau)\big(W_{1,0}f'\big)(\tau)d\tau+\iint_D \varphi_3(\tau,\xi)\big(W_{1,1}f'\big)(\tau,\xi)d\tau d\xi, \qquad (10)$$

where $\varphi_0 = z(t_0,x_0)$, $\varphi_1(\xi) = z_\xi(t_0,\xi)$, $\varphi_2(\tau) = z_\tau(\tau,x_0)$, $\varphi_3(\tau,\xi) = z_{\tau\xi}(\tau,\xi)$, $f' = (f_0',f_1',f_2',f_3')$ (we denoted f' of four column vectors accordingly), operators $W_{0,0}, W_{0,1}, W_{1,0}, W_{1,1}$ are defined as follows

$$W_{0,0}f' \equiv f_0' + \int_T \sum_{j=1}^m \beta_j(t)f_2'(t)dt + \iint_D A_0(t,x)f_3'(t,x)dtdx,$$

$$\big(W_{0,1}f'\big)(\xi) \equiv f_1'(\xi) + \int_T \sum_{j=1}^m \theta(\xi_j-\xi)\beta_j(t)\, f_2'(t)dt +$$

$$+\iint_D \theta(x-\xi)A_0(t,x)f_3'(t,x)dtdx + \int_T A_1(t,\xi)f_3'(t,\xi)dt, \ \ \xi\in X$$

$$\big(W_{1,0}f'\big)(\tau) \equiv \sum_{j=1}^m \alpha_j(\tau)f_2'(\tau) + \int_T \theta(t-\tau)\sum_{j=1}^m \beta_j(t)f_2'(t)dt +$$

$$+\iint_D \theta(t-\tau)A_0(t,x)f_3'(t,x)dtdx + \int_T A_2(\tau,x)f_3'(\tau,x)dx +$$

$$+\iint_D K(\tau,t,x)f_3'(t,x)dtdx, \quad \tau\in T, \qquad (11)$$

$$\big(W_{1,1}f'\big)(\tau,\xi) \equiv f_3'(\tau,\xi) + \int_T \theta(t-\tau)\sum_{j=1}^m \theta(\xi_j-\xi)\beta_j(t)\, f_2'(t)dt +$$

$$+\sum_{j=1}^m \theta(\xi_j-\xi)\alpha_j(\tau)f_2'(\tau) + \iint_D \theta(t-\tau)\theta(x-\xi)\, A_0(t,x)f_3'(t,x)dtdx +$$

$$+\int_T \theta(t-\tau)A_1(t,\xi)f_3'(t,\xi)dt + \int_X \theta(x-\xi)\, A_2(\tau,x)f_3'(\tau,x)dx +$$

$$+\iint_D \theta(h(t,x)-\xi)K(\tau;t,x)f_3'(t,x)dtdx, \quad (\tau,\xi)\in D,$$

and $\theta(t)$ is the Heaviside function.

Now we shall introduce the concept of an adjoint problem.

Definition 1. We shall call adjoint problem the following system of column vector equations with respect to desired four $f = (f_0,f_1,f_2,f_3)\in\Delta_{q,n}(D)$:

$$W_{0,0}f' = \gamma_0';\ \big(W_{0,1}f'\big)(\xi) = \gamma_1'(\xi), \quad \xi\in X;$$

$$\big(W_{1,0}f'\big)(\tau) = \gamma_2'(\tau), \ \ \tau\in T; \quad \big(W_{1,1}f'\big)(\tau,\xi) = \gamma_3'(\tau,\xi), \quad (\tau,\xi)\in D. \quad (12)$$

Here $\gamma_0\in R^n$, $\gamma_1\in L_{q,n}(X)$, $\gamma_2\in L_{q,n}(T)$, $\gamma_3\in L_{q,n}(D)$ are some line $n-$ vectors. We will also call the operator $W = \big(W_{0,0},W_{0,1},W_{1,0},W_{1,1}\big)$ the adjoint operator to the operator $V = \big(V_{0,0},V_{0,1},V_{1,0},V_{1,1}\big)$, i.e., $W = V^*$. This operator W acts in the space

$$\Delta'_{q,n}(D) = (R^n)' \times L'_{q,n}(X) \times L'_{q,n}(T) \times L'_{q,n}(D),$$

where $(R^n)'$ is the space of all numerical column $n-$ vectors, $L'_{q,n}(X), L'_{q,n}(T), L'_{q,n}(D)$ are the spaces of all $n-$ dimentional column vector-functions with elements from $L_q(X), L_q(T), L_q(D),$ respectively.

Note that considering the views (11) of the operators $W_{0,0}, W_{0,1}, W_{1,0}, W_{1,1}$ we may say that the adjoint problem (12) is an inteqro -algebraic system with respect to the desired four $f' = (f'_0, f'_1, f'_2, f'_3) \in \Delta'_{q,n}(D).$

5. THE SYSTEM OF MATRIX EQUATIONS

We will use a concept of the space $\Delta_{q,n\times n}(D) = R^{n\times n} \times L_{q,n\times n}(X) \times L_{q,n\times n}(T) \times L_{q,n\times n}(D)$ of the fours $F = (F_0, F_1, F_2, F_3)$ of all $n \times n-$ matrices $F_0, F_1(x), F_2(t), F_3(t,x).$ Here $R^{n\times n}$ is the space of numerical $n \times n-$ matrices; $L_{q,n\times n}(X),$ $L_{q,n\times n}(T),$ $L_{q,n\times n}(D)$ are the spaces of $n \times n-$ matrices $F_1(x), F_2(t), F_3(t,x)$ with elements from $L_q(X), L_q(T), L_q(D),$ respectively.

Now introduce following system of matrix equations with respect to the desired four $F = \big(F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi)\big)$ of matrices $F_0 \in R^{n\times n}, F_1 \in L_{q,n\times n}(X),$ $F_2 \in L_{q,n\times n}(T),$ $F_3 \in L_{q,n\times n}(D)$:

$$\overline{W}_{0,0}F = \Gamma_0; \qquad \big(\overline{W}_{0,1}F\big)(\xi) = \Gamma_1(\xi), \quad \xi \in X;$$

$$\big(\overline{W}_{1,0}F\big)(\tau) = \Gamma_2(\tau), \ \ \tau \in T; \ \big(\overline{W}_{1,1}F\big)(\tau,\xi) = \Gamma_3(\tau,\xi), \quad (\tau,\xi) \in D. \qquad (13)$$

Here $\Gamma_0,$ $\Gamma_1(\xi),$ $\Gamma_2(\tau),$ $\Gamma_3(\tau,\xi)$ are some $n \times n-$ matrices from $R^{n\times n},$ $L_{q,n\times n}(X),$ $L_{q,n\times n}(T),$ $L_{q,n\times n}(D),$ respectively; $F = (F_0, F_1, F_2, F_3)$ is desired four of $n \times n-$ matrices $F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi);$ $\overline{W}_{0,0}, \overline{W}_{0,1}, \overline{W}_{1,0}, \overline{W}_{1,1}$ are matrix operators, which are expressed in terms of the operators (11) as follows:

$$\overline{W}_{0,0} = \big(W_{0,0}F^1, \dots, W_{0,0}F^n\big),$$

$$\big(\overline{W}_{0,1}\big)(\xi) = \Big(\big(W_{0,1}F^1\big)(\xi), \dots, \big(W_{0,1}F^n\big)(\xi)\Big),$$

$$\big(\overline{W}_{1,0}\big)(\tau) = \Big(\big(W_{1,0}F^1\big)(\tau), \dots, \big(W_{1,0}F^n\big)(\tau)\Big), \qquad (14)$$

$$\big(\overline{W}_{1,1}\big)(\tau,\xi) = \Big(\big(W_{1,1}F^1\big)(\tau,\xi), \dots, \big(W_{1,1}F^n\big)(\tau,\xi)\Big).$$

where $F^j = \Big(F_0^j, F_1^j(\xi), F_2^j(\tau), F_3^j(\tau,\xi)\Big)$ and $F_0^j, F_1^j(\xi), F_2^j(\tau), F_3^j(\tau,\xi)$ are $j-$ th $(j = 1, \dots, n)$ columns of the matrices $F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi),$ respectively.

It seems that each $j-$ th $(j = 1, \dots, n)$ four $F^j = \Big(F_0^j, F_1^j(\xi), F_2^j(\tau), F_3^j(\tau,\xi)\Big)$ of $j-$ th columns $F_0^j, F_1^j(\xi), F_2^j(\tau), F_3^j(\tau,\xi)$ of the corresponding matrices $F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi)$ is a solution of the adjoint problem (12) with right-hand sides

$$\gamma'_0 = \Gamma_0^j, \qquad \gamma'_1(\xi) = \Gamma_1^j(\xi),$$

$$\gamma'_2(\tau) = \Gamma_2^j(\tau), \qquad \gamma'_3(\tau,\xi) = \Gamma_3^j(\tau,\xi).$$

In other words, every four $F^j = \Big(F_0^j, F_1^j(\xi), F_2^j(\tau), F_3^j(\tau,\xi)\Big)$ is a solution of the following integro-algebraic system:

$$W_{0,0}F^j = \Gamma_0^j; \qquad \big(W_{0,1}F^j\big)(\xi) = \Gamma_1^j(\xi), \ \ \xi \in X;$$

$$\big(W_{1,0}F^j\big)(\tau) = \Gamma_2^j(\tau), \ \ \tau \in T; \quad \big(W_{1,1}F^j\big)(\tau,\xi) = \Gamma_3^j(\tau,\xi), \ \ (\tau,\xi) \in D, \qquad (15)$$

where $\Gamma_0^j, \Gamma_1^j(\xi), \Gamma_2^j(\tau), \Gamma_3^j(\tau,\xi)$ are $j-$ th columns of the corresponding matrices $\Gamma_0, \Gamma_1(\xi), \Gamma_2(\tau), \Gamma_3(\tau,\xi).$

Last system (15) is the same system (12) in which $f' = F^j,$ $f'_0 = F_0^j,$ $f'_1(\xi) = F_1^j(\xi),$ $f'_2(\tau) = F_2^j(\tau),$ $f'_3(\tau,\xi) = F_3^j(\tau,\xi).$ Thus, the solution $F = \big(F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi)\big)$ of the system of matrix equations (13) consists of a list of the solutions $F^j = \left(F_0^j, F_1^j(\xi), F_2^j(\tau), F_3^j(\tau,\xi)\right)$ of the adjoint problem (12). Basing on linearity of the operators (11) and having expressions (14) of the operators $\overline{W}_{0,0}, \overline{W}_{0,1}, \overline{W}_{1,0}, \overline{W}_{1,1}$ by the operators $W_{0,0}, W_{0,1}, W_{1,0}, W_{1,1}$ we can write the system of matrix equations (13) as follows

$$W_{0,0}F = \Gamma_0; \qquad \left(W_{0,1}F\right)(\xi) = \Gamma_1(\xi), \quad \xi \in X;$$

$$\left(W_{1,0}F\right)(\tau) = \Gamma_2(\tau), \quad \tau \in T; \quad \left(W_{1,1}F\right)(\tau,\xi) = \Gamma_3(\tau,\xi), \quad (\tau,\xi) \in D. \quad (16)$$

The system of matrix equations (13) in this form unites the systems (15) for all $j = 1, \dots, n$, i.e., it is enuf to write in the system (12) $f' = F$, $\gamma_0' = \Gamma_0$, $\gamma_1'(\xi) = \Gamma_1(\xi)$, $\gamma_2'(\tau) = \Gamma_2(\tau)$, $\gamma_3'(\tau,\xi) = \Gamma_3(\tau,\xi)$.

Basing on abovementioned and paying attention to the expressions of the matrix operators $\overline{W}_{0,0}, \overline{W}_{0,1}, \overline{W}_{1,0}, \overline{W}_{1,1}$, we may say that following theorem is truth.

Theorem 1. If the adjoint problem (12) for each right-hand side $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in \Delta_{q,n}(D)$ has a solution $f = (f_0, f_1, f_2, f_3) \in \Delta_{q,n}(D)$, then the system of matrix equations (13) or (16) also for each right-hand side

$$\Gamma = (\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3) \in \Delta_{q,n\times n}(D)$$

has a solution $F = (F_0, F_1, F_2, F_3) \in \Delta_{q,n\times n}(D)$.

Proof. As we noted above the operators $\overline{W}_{0,0}, \overline{W}_{0,1}, \overline{W}_{1,0}, \overline{W}_{1,1}$ of the system of matrix equations (13) are expressed by the operators $W_{0,0}, W_{0,1}, W_{1,0}, W_{1,1}$ of adjoint problem (12) as have been shown in (14). Further, there had been shown that the system of matrix equations (13) involves a list of n adjoint problems (12) which were represented as the systems (15) with right-hand sides

$$\gamma_0' = \Gamma_0^j, \qquad \gamma_1'(\xi) = \Gamma_1^j(\xi),$$

$$\gamma_2'(\tau) = \Gamma_2^j(\tau), \quad \gamma_3'(\tau,\xi) = \Gamma_3^j(\tau,\xi), \quad j = 1, \dots, n.$$

Then, the proof of the theorem is become clear.

Besides Theorem 1 we shall note also the following important theorem [14 – 16].

Theorem 2. Let adjoint problem (12) for zero right-hand sides, i.e.,

$$W_{0,0}f' = 0; \qquad \left(W_{0,1}f'\right)(\xi) = 0, \quad \xi \in X;$$

$$\left(W_{1,0}f'\right)(\tau) = 0, \quad \tau \in T; \qquad \left(W_{1,1}f'\right)(\tau,\xi) = 0, \quad (\tau,\xi) \in D,$$

has unique trivial zero solution, i.e., for solution $f = \left(f_0, f_1(\xi), f_2(\tau), f_3(\tau,\xi)\right)$, the components are equal zero:

$$f_0 = 0, \qquad f_1(\xi) \equiv 0, \quad \xi \in X;$$

$$f_2(\tau) \equiv 0, \quad \tau \in T; \qquad f_3(\tau,\xi) \equiv 0, \quad (\tau,\xi) \in D.$$

Then the adjoint problem (12) for each given four $\gamma_0 \in R^n$, $\gamma_1 \in L_{q,n}(X)$, $\gamma_2 \in L_{q,n}(T)$, $\gamma_3 \in L_{q,n}(D)$ of the right-hand sides has a unique solution $f = \left(f_0, f_1(\xi), f_2(\tau), f_3(\tau,\xi)\right)$ from the space

$$\Delta_{q,n}(D) = R^n \times L_{q,n}(X) \times L_{q,n}(T) \times L_{q,n}(D).$$

Here is also true the following theorem.

Theorem 3. Let adjoint problem (12) with zero right-hand sides

$$W_{0,0}f' = 0; \ \left(W_{0,1}f'\right)(\xi) = 0, \quad \xi \in X;$$

$$\left(W_{1,0}f'\right)(\tau) = 0, \quad \tau \in T; \quad \left(W_{1,1}f'\right)(\tau,\xi) = 0, \quad (\tau,\xi) \in D$$

has unique trivial zero solution, i.e., for solution $f = \left(f_0, f_1(\xi), f_2(\tau), f_3(\tau,\xi)\right)$ the components are equal zero:

$f_0 = 0, \qquad f_1(\xi) \equiv 0, \; \xi \in X;$

$$f_2(\tau) \equiv 0, \; \tau \in T; \qquad f_3(\tau,\xi) \equiv 0, \; (\tau,\xi) \in D.$$

Then the system of matrix equations (13) or (16) for each given four $\Gamma_0 \in R^{n\times n}$, $\Gamma_1 \in L_{q,n\times n}(X)$, $\Gamma_2 \in L_{q,n\times n}(T)$, $\Gamma_3 \in L_{q,n\times n}(D)$ of right-hand sides will have unique solution $F = \left(F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi)\right)$ from the space

$$\Delta_{q,n\times n}(D) = R^{n\times n} \times L_{q,n\times n}(X) \times L_{q,n\times n}(T) \times L_{q,n\times n}(D).$$

Proof. Based on the theorem 1 and using the theorem 2 we may say that if the adjoint problem (12) has a unique solution for any right-hand sides, then system of matrix equation (16) has a unique solution for any right-hand sides.

We can get the kind (16) of the system of matrix equations (13) by putting in the system (12) $f' = F$, $f_0' = F_0$, $f_1'(\xi) = F_1(\xi)$, $f_2'(\tau) = F_2(\tau)$, $f_3'(\tau,\xi) = F_3(\tau,\xi)$ and $\gamma_0' = \Gamma_0$, $\gamma_1'(\xi) = \Gamma_1(\xi)$, $\gamma_2'(\tau) = \Gamma_2(\tau)$, $\gamma_3'(\tau,\xi) = \Gamma_3(\tau,\xi)$. In this case, we will also take these values into account in expressions (9) and (10). Then we will get the following vector relation:

$$\varphi_0 \cdot W_{0,0}F + \int_X \varphi_1(\xi)(W_{0,1}F)(\xi)d\xi +$$

$$+ \int_T \varphi_2(\tau)\left(W_{1,0}F\right)(\tau)d\tau + \iint_D \varphi_3(\tau,\xi)\left(W_{1,1}F\right)(\tau,\xi)d\tau d\xi =$$

$$= \left(V_{0,0}z\right)F_0 + \int_X \left(V_{0,1}z\right)(\xi)F_1(\xi)d\xi +$$

$$+ \int_T \left(V_{1,0}z\right)(\tau)F_2(\tau)d\tau + \iint_D \left(V_{1,1}z\right)(\tau,\xi)F_3(\tau,\xi)d\tau d\xi. \qquad (17)$$

Remind that

$$\varphi_0 = z(t_0, x_0), \qquad \varphi_1(\xi) = z_\xi(t_0,\xi),$$

$$\varphi_2(\tau) = z_\tau(\tau, x_0), \qquad \varphi_3(\tau,\xi) = z_{\tau\xi}(\tau,\xi).$$

Note that for the solution $z \in W_{p,n}(D)$ of the problem (1)-(4) with the right-hand sides $g_0 \in R^n$, $g_1 \in L_{p,n}(X)$, $g_2 \in L_{p,n}(T)$, $g_3 \in L_{p,n}(D)$, and for the solution $F = (F_0, F_1, F_2, F_3) \in \Delta_{q,n\times n}(D) = R^{n\times n} \times L_{q,n\times n}(X) \times L_{q,n\times n}(T) \times L_{q,n\times n}(D)$ of the problem (16) with the right-hand sides $\Gamma_0 \in R^{n\times n}$, $\Gamma_1 \in L_{q,n\times n}(X)$, $\Gamma_2 \in L_{q,n\times n}(T)$, $\Gamma_3 \in L_{q,n\times n}(D)$ the relation (17) will be in the following view:

$$\varphi_0 \cdot \Gamma_0 + \int_X \varphi_1(\xi)\cdot\Gamma_1(\xi)d\xi + \int_T \varphi_2(\tau)\cdot\Gamma_2(\tau)d\tau +$$

$$+ \iint_D \varphi_3(\tau,\xi)\Gamma_3(\tau,\xi)d\tau d\xi = g_0 \cdot \mathrm{F}_0 + \int_X g_1(\xi)\cdot \mathrm{F}_1(\xi)d\xi +$$

$$+ \int_T g_2(\tau)\cdot \mathrm{F}_2(\tau)d\tau + \iint_D g_3(\tau,\xi)\mathrm{F}_3(\tau,\xi)d\tau d\xi. \qquad (18)$$

Now we will consider a special case of the matrix equations system (16) for the right-hand sides $\Gamma_0 = E$, $\Gamma_1(\xi) = \theta(x-\xi)\cdot E$, $\Gamma_2(\tau) = \theta(t-\tau)\cdot E$, $\Gamma_3(\tau,\xi) = \theta(t-\tau)\theta(x-\xi)\cdot E$ with parameters $t \in T$ and $x \in X$:

$$W_{0,0}F = E; \quad (W_{0,1}F)(\xi) = \theta(x-\xi)\cdot E, \quad \xi \in X;$$

$$(W_{1,0}F)(\tau) = \theta(t-\tau)\cdot E, \quad \tau \in T;$$

$$(W_{1,1}F)(\tau,\xi) = \theta(t-\tau)\theta(x-\xi)\cdot E, \quad (\tau,\xi) \in D, \tag{19}$$

where E is the unit $n \times n$ matrix and $\theta(t)$ is the Heaviside function.

6. FUNDAMENTAL SOLUTION AND INTEGRAL REPRESENTATION

Now we will introduce a concept of the fundamental solution for the linear nonlocal boundary value problem (1)-(4).

Definition 2. The solution $F = (F_0, F_1, F_2, F_3) \in \Delta_{q,n\times n}(D)$ of the system of matrix equations (19) with right-hand sides in special view will be called fundamental solution for the problem (1)-(4) and denoted $\tilde{F}(t,x) = \left(\tilde{F}_0(t,x), \tilde{F}_1(t,x;\cdot), \tilde{F}_2(t,x;\cdot), \tilde{F}_3(t,x;\cdot,\ \cdot)\right) \in \Delta_{q,n\times n}(D)$. It's clear that the fundamental solution in addition to independent working variables $(\tau,\xi) \in D,$ also depends on the parameters $(t,x) \in D$.
In connection with introduced concept of the fundamental solution we shall note the following theorem.

Theorem 4. Let the adjoint problem (12) with zero right-hand sides

$$W_{0,0}f' = 0; \qquad (W_{0,1}f')(\xi) = 0, \ \xi \in X;$$
$$(W_{1,0}f')(\tau) = 0, \ \tau \in T; \qquad (W_{1,1}f')(\tau,\xi) = 0, \ (\tau,\xi) \in D,$$

has unique trivial zero solution $f = (0,0,0,0) \in \Delta_{q,n}(D)$ i.e.,

$$f_0 = 0, \qquad f_1(\xi) \equiv 0, \ \xi \in X;$$
$$f_2(\tau) \equiv 0, \quad \tau \in T; \qquad f_3(\tau,\xi) \equiv 0, \quad (\tau,\xi) \in D.$$

Then the system of matrix equations (19) has a unique solution

$$\tilde{F}(t,x) = \left(\tilde{F}_0(t,x), \tilde{F}_1(t,x;\cdot), \tilde{F}_2(t,x;\cdot), \tilde{F}_3(t,x;\cdot,\cdot)\right)$$

from the space

$$\Delta_{q,n\times n}(D) = R^{n\times n} \times L_{q,n\times n}(X) \times L_{q,n\times n}(T) \times L_{q,n\times n}(D).$$

Proof. Basing on the theorem 3 we get of proof of this theorem. Indeed, the special system of matrix equations (19) with parameters $(t,x) \in D$ for almost every $(t,x) \in D$ is partial case of the matrix equations system (16) with respect to desired four $F = \left(F_0, F_1(\xi), F_2(\tau), F_3(\tau,\xi)\right)$. According to the abovementioned we are convinced of the proof of the theorem.

Thus, for the fundamental solution, i.e., for the solution of the matrix equations system (16) corresponding to the right-hand sides

$$\Gamma_0 = E, \ \Gamma_1(\xi) = \theta(x-\xi)\cdot E, \ \Gamma_2(\tau) = \theta(t-\tau)\cdot E,$$

$$\Gamma_3(\tau,\xi) = \theta(t-\tau)\theta(x-\xi)\cdot E, \tag{20}$$

the relation (18) will be as follows

$$\varphi_0 + \int_{x_0}^{x} \varphi_1(\xi)\,d\xi + \int_{t_0}^{t} \varphi_2(\tau)\,d\tau + \int_{t_0}^{t}\int_{x_0}^{x} \varphi_3(\tau,\xi)d\tau d\xi =$$

$$= g_0\tilde{F}_0(t,x) + \int_X g_1(\xi)\,\tilde{F}_1(t,x;\xi)d\xi +$$

$$+ \int_T g_2(\tau)\tilde{F}_2(t,x;\tau)\,d\tau + \iint_D g_3(\tau,\xi)\,\tilde{F}_3(t,x;\tau,\xi)d\tau d\xi. \tag{21}$$

Taking the expression (5) into account in the last relation (21) we get the following representation of the solution $z \in W_{p,n}(D)$ of the linear nonlocal boundary value problem (1)-(4):

$$z(t,x) = g_0\tilde{F}_0(t,x) + \int_X g_1(\xi)\tilde{F}_1(t,x;\xi)d\xi +$$

$$+\int_T g_2(\tau)\tilde{F}_2(t,x;\tau)\,d\tau + \iint_D g_3(\tau,\xi)\tilde{F}_3(t,x;\tau,\xi)\,d\tau d\xi. \qquad (22)$$

Finally, let us note the following last theorem.

Theorem 5. Let the adjoint problem (12) with zero right-hand sides has unique trivial zero solution $f = (0,0,0,0) \in \Delta_{q,n}(D)$. Then the nonlocal linear boundary value problem (1)-(4) has the fundamental solution

$$\tilde{F}(t,x) = \left(\tilde{F}_0(t,x), \tilde{F}_1(t,x;\cdot), \tilde{F}_2(t,x;\cdot), \tilde{F}_3(t,x;\cdot,\cdot)\right)$$

from the space

$$\Delta_{q,n\times n}(D) = R^{n\times n} \times L_{q,n\times n}(X) \times L_{q,n\times n}(T) \times L_{q,n\times n}(D)$$

for almost every $(t,x) \in D$ and there takes plase the integral representation (22) for the solution $z \in W_{p,n}(D)$ of the problem (1)-(4).

Proof. First part of the proof of this theorem follows from the theorem 4. Further, if the solution $z \in W_{p,n}(D)$ of the nonlocal problem (1)-(4) has fundamental solution then as we noted above there takes plase inteqral representation (22) for this solution $z(t,x)$.

Remind that the concept of the fundamental solution introduced for the considering linear nonlocal problem (1)-(4) is more general in comparison with the Green's function and the Riemann function.

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