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# About the Solution of One Bondary Value Problems

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**Abstract.** In this article, the conditions for the regular solution of the boundary problem for a class of second-order operator-differential equations with truncated coefficients are derived. These equations have been specifically constructed to mathematically determine the corrosion time of metals in aggressive environments, which has crucial implications in fields such as materials science and engineering. The study focuses on identifying the precise conditions under which the problem exhibits regular solvability, a key aspect of ensuring the reliability of the solutions in practical applications. Furthermore, the obtained conditions are explicitly expressed in terms of the operator coefficients of the equation, offering a more robust framework for understanding the solvability of such operator-differential equations in various contexts. These results contribute significantly to both the theoretical and applied aspects of mathematical physics and engineering, especially in the analysis of physical systems subject to corrosion and other environmental factors.

**Keywords:** Boundary value problem, Differential equation, Hilbert space, Intermediate derivatives, Regular solvability.

## 1. INTRODUCTION

Many issues in mechanics, mathematical physics, and the theory of partial differential equations necessitate a comprehensive examination of the solvability of boundary value problems for operator-differential equations across various functional spaces [1-7]. These investigations are essential not only for theoretical understanding but also for practical applications in engineering, physics, and applied mathematics.

For instance, certain challenges in the theory of elasticity within strips [8-10], along with problems concerning the vibrations of mechanical systems [9] and the oscillations of elastic cylinders [10], underscore the critical nature of studying specific boundary value problems associated with operator-differential equations. Such studies are foundational for advancing the spectral theory of quadratic beams and higher-order beams, which play a pivotal role in modern structural analysis and design.

An illustrative example of this relationship is the analysis of the stress-strain state of a slab, which necessitates the resolution of problems related to the theory of elasticity in strips. This exploration not only enhances our theoretical frameworks but also informs practical design considerations in civil and mechanical engineering, where understanding material behavior under stress is paramount. The ability to predict how structures will respond to external forces is vital for ensuring safety and reliability in engineering projects.

In addition, the quest for precise values of norms or their upper bounds for operators of intermediate derivatives presents significant mathematical interest. These investigations are critical for developing theoretical insights and practical methodologies in various fields of mathematical analysis [11,12]. For example, in approximation theory [13], understanding these norms facilitates the creation of better approximation schemes, which are vital for numerical analysis and computational methods. This can lead to more accurate simulations in engineering applications, improving design processes and outcomes.

Moreover, the interplay between theoretical advances and practical applications highlights the importance of interdisciplinary research. The methodologies developed through studying operator-differential equations often find applications in diverse areas such as control theory, signal processing, and even financial mathematics. For instance, in control systems, understanding the dynamics of operator-differential equations can lead to more effective control strategies, optimizing system performance. Similarly, in financial mathematics, the techniques derived from these studies can be used to model complex financial instruments, providing better risk assessments and decision-making tools.

Ultimately, the ongoing research in this area promises to deepen our understanding of both fundamental and applied problems, paving the way for new discoveries and advancements in science and engineering. The continuous exploration of these topics not only enriches the mathematical landscape but also drives innovation, opening new avenues for research and practical applications across multiple disciplines.

## 2. SOME DEFINITION AND AUXILIARY FAKTS

Let $H$ is a separable Hilbert space, $A$ positive definite self-adjoint operator in $H$ with domain $D(A)$. Let


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$\gamma \geq 0$. Then, as is known, the domain of definition of the operator $A^{\gamma}$ turns into a Hilbert space $H_{\gamma}$ i.e.

$$H_{\gamma} = D(A^{\gamma}),(x,y)_{\gamma} = (A^{\gamma}x, A^{\gamma}y), x, y \in H_{\gamma}.$$

At $\gamma = 0$ believe that $H_0 = H$, but $(x,y)_0 = (x,y)$.

Let $a$ and $b$ be real numbers such that $-\infty \leq a < b \leq \infty$. We denote by $L_2((a,b);H)$ the Hilbert space of all vector functions defined on an interval $(a,b)$ almost everywhere, with values in $H$, measurable and square integrable with the norm

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f(t)\|^2 dt\right)^{1/2} < \infty.$$

Obviously, the scalar multiplication in $L_2((a,b);H)$ is given by the formula

$$(f,g)_{L_2((a,b);H)} = \int_a^b (f(t), g(t)) dt, \quad f(t), g(t) \in L_2((a,b);H).$$

Assumme $a = -\infty$, $b = -\infty$, i.e. $(-\infty\infty) = R$ believe that $L_2((a,b);H) = L_2((-\infty,\infty);H) = L_2(R;H)$, but $a = 0$, $b = -\infty$, i.e. $(a,b) = (0,\infty) = R_+$. Believe that $L_2((a,b);H) == L_2((0,\infty);H) = L_2(R_+;H)$.

Following the monograph [1] we introduce Hilbert spaces for $m \geq 1$ ($m$-natural number) $W_2^m((a,b);H) = \{u(t): u^{(m)}(t) \in L_2((a,b);H), A^m u(t) \in L_2((a,b);H)\}$ the scalar multiplication $(u,v)_{W_2^m((a,b);H)} = \int_a^b (u^{(m)}(t), v^{(m)}(t)) dt + \int_a^b (A^m u(t), A^m v(t)) dt$.

Here and in what follows, we will understand all derivatives in space $H$ the sense of distribution theory [1]. For this work, we use spaces $W_2^m((a,b);H)$ at $m = 1, m = 2$.

Here, we also assume that for $(a,b) = R = (-\infty,\infty)$

$$W_2^m((-\infty,\infty);H) = W_2^m(R;H),$$

but $a = 0, b = \infty$ $((a,b) = R_+ = (0,\infty))$

$$W_2^m((0,\infty);H) = W_2^m(R_+;H).$$

Let us note some properties of the space $W_2^m((a,b);H)$ [1]:

1.If $u \in W_2^m(R_+;H)$ then there are numbers $c_k > 0$

$$\left\| A^{m-k} \frac{d^k u}{dt^k} \right\|_{L_2((a,b);H)} \leq c_k \|u\|_{W_2^m((a,b);H)}, k = \overline{0,m}.$$

This is called the intermediate derivative theorem.

2.If $u \in W_2^m(R_+;H)$ then the operator $L_k u = u^{(k)}(t_0)$ is a continuous operator from the space $W_2^m((a,b);H)$ in $H_{m-k-1/2}, k = \overline{0,m-1}$, i.e.

$$\left\| A^{m-k-\frac{1}{2}} u^{(k)}(t_0) \right\|_{L_2((a,b);H)} \leq \beta_k \|u\|_{W_2^m((a,b);H)}, k = \overline{0,m-1},$$

where $t_0 \in [a,b]$. When researching local issues, we need to find the following subspaces $W_2^2((a,b);H)$

$$W_2^2(a,b;0,1) = \{u : u \in W_2^2((a,b);H), u(a) = 0, u(b) = 0\},$$

$$W_2^2(a,b;0,1) = \{u : u \in W_2^2((a,b);H), u'(a) - u'(b) = 0\},$$

$$W_2^2(a,b;\alpha) = \{u : u \in W_2^2((a,b);H), u(0) = e^{ia}u(b), u'(0) = e^{ib}u'(b)\},$$

as well as sub-space $W_2^1((a,b);H)$:

$$W_2^1(a,b;0,1) = \{u : u \in W_2^1((a,b);H), u(a) = u(b) = 0\}.$$

Note that if $u \in W_2^2(R;H)$, $u \in W_2^2((a,b);H)$ then $\|u\|_{W_2^2((a,b);H)} \leq const \|u\|_{W_2^2(R;H)}$ Here $-\infty < a < b < \infty$.

This fact flows from the theorem on continuation [1]. At $\varphi \in H_{3/2}, e^{-At}\varphi \in W_2^2((a,b);H), 0 \le a < b < \infty$. Really,

$$\left\| e^{-At}\varphi \right\|^2_{W_2^2(R;H)} = \left\| A^2 e^{-At}\varphi \right\|^2_{L_2((a,b);H)} + \left\| A^2 e^{-At}\varphi \right\|^2_{L_2((a,b);H)} =$$

$$= 2\left\| A^2 e^{-At}\varphi \right\|^2_{L_2((a,b);H)} = 2\int_0^\infty \left(A^2 e^{-At}\varphi, A^2 e^{-At}\varphi\right) dt$$

Denote by $A^{3/2}\varphi = \psi$ . Then

$$\left\| e^{-At}\varphi \right\|^2_{W_2^2(R;H)} = 2\int_0^\infty \left( A^{1/2} e^{-At}\psi, A^{1/2} e^{-At}\psi \right) dt = 2\int_0^\infty \left( A e^{-2At}\psi, \psi \right) dt = 2\int_a^b \int_{\mu_0}^\infty \left( \sigma e^{-2\sigma t}\psi, \psi \right) d\left(E_\theta \psi, \psi\right) =$$

$$= 2\sigma \int_{\mu_0}^\infty \left( \sigma \int_a^b e^{-2\sigma t} \right) d\left(E_\theta \psi, \psi\right) = \int_{\mu_0}^\infty e^{-2\sigma t} \Big|_a^b \left(d\, E_\theta \psi, \psi\right) = e^{-2a\mu_0}\left(1 - e^{-(b-a)\mu_0}\right) \left\| \varphi \right\|^2_{3/2} .$$

Thus, $e^{-At}\varphi \in W_2^2((a,b);H)$, at $\varphi \in H_{3/2}$. Conversely, if $e^{-At}\varphi \in W_2^2(R_+;H)$, then it follows from the trace theorem that $\varphi \in H_{3/2}$. Next, we note that if $a = 0, b = T > 0$, then the general solution to the equation $-u\ \ (t) + A^2 u(t) = 0, t \in (0,t)$ presented in the form
$u_0(t) = e^{-tA}\varphi_0 + e^{-(t-T)A}\varphi_1, \varphi_0, \varphi_1 \in H_{3/2}$. Similarly $\varphi \in H_{1/2}$, then $e^{-At}\varphi \in W_2^2((a,b);H)$, $0 \le a < b < \infty$.

Really $\left\| e^{-At}\varphi \right\|^2_{W_2^2((a,b);H)} = 2\left\| A^{1/2} e^{-At}\varphi \right\|^2_{L_2((a,b);H)}$. Assuming $A^{1/2}\varphi = \psi$ have

$$\left\| e^{-At}\varphi \right\|^2_{W_2^2((a,b);H)} = 2\left\| A^{1/2} e^{-At}\varphi \right\|^2_{L_2((a,b);H)} = e^{-2a\mu_0}\left(1 - e^{-2(b-a)\mu_0}\right) \left\| \varphi \right\|_{1/2} .$$

At $a = 0, b = \infty$ we obtain $\left\| e^{-At}\varphi \right\|^2_{W_2^2((a,b);H)} \le \left\| \varphi \right\|_{3/2}$. Further, note that the general solution of the equation

$$-\frac{d^2 u(t)}{dt^2} + f(t) A^2 u(t) = 0$$

from the space $W_2^2((0,T);H)$ has the form

$$u_0(t) = \begin{cases} e^{-\alpha t A}\varphi_1 + e^{-\alpha(t_0 - t)A}\varphi_2, t \in (0, t_0) \\ e^{-\beta(t - t_0)A}\varphi_3 + e^{\beta(T - t)A}\varphi_4, t \in (t_0, T) \end{cases}$$

where $A$ a positive definite self-adjoint operator,

$$u_0(t) = \begin{cases} \alpha^2, t \in (0, t_0) \\ \beta^2, t \in (t_0, T), t_0 \in (0, T) \end{cases}$$

$\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4 \in H_{3/2}$ and the equalities must be satisfied

$$u(t_0 - 0) = u(t_0 + 0)$$
$$u'(t_0 - 0) = u'(t_0 + 0).$$

Further, we note that $L(X,Y)$ we will denote the space of linear bounded operators acting from the space $X$ to the Banach space $Y$ . Through $\sigma_\infty(H)\, \sigma_\infty(H)$ denote the space of completely continuous operators acting in $H$
.

Now let's enter the space below

$$\overset{0}{W}{}^{2}_{2,S} \left\{ u : u \in W_2^2((0,T);T), u\ (0) = Su'(0), u\ (T) = 0 \right\}$$

where $S \in L\left(H_{1/2}, H_{3/2}\right)$.

## 3. ON THE EXİSTENCE OF REGULAR SOLUTİON OF BOUNDARY VALUE PROBLEMS

Consider in a separable Hilbert space $H$ boundary value problems

$$-\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t)+A_1(t)\frac{du(t)}{dt}+A_2(t)u(t)=f(t) \qquad (1)$$

$$u(0)=Su'(0), u(T)=0, \qquad (2)$$

where $u(t), f(t)$ vector function defined in $(0,T)$ almost everywhere with values in $H$, the operator coefficients satisfy the conditions:

1) $A$ positive definite self-adjoint operator in $H$;
2) $\rho(t)$ numeric function

$$\rho(t)=\begin{cases}\alpha^2, t\in(0,t_0)\\ \beta^2, t\in(t_0,T), t_0\in(0,T)\end{cases}$$

where $\alpha>0, \beta>0$;

3) operators $A_1(t)$, $A_2(t)$ at each $t\in(0,T)$ linear; $A_1(t)A^{-1}$, $A_2(t)A^{-2}$ are restricted $H$

$$\sup_t\left\|A_1(t)A^{-1}\right\|<\infty,\ t\in(0,T),\ \sup_t\left\|A_2(t)A^{-2}\right\|<\infty,\ t\in(0,T);$$

4) operator $S:H_{3/2}\to H_{1/2}$ linear, $S\in L\left(H_{1/2},H_{3/2}\right)$.

**Definition 1**. If at $f(t)\in L_2((0,T);H)$ exists a vector function $u(t)\in W_2^2(R_+;H)$, satisfying equation (1) almost everywhere, then we will call it a regular solution of equation (1).

**Definition 2**. If at $f(t)\in L_2((0,T);H)$ there is a regular solution $u(t)\in W_2^2(R_+;H)$ equation (1), which satisfies the boundary conditions (2) in the sense of convergence

$$\lim_{t\to t+0}\left\|u(t)-Su(t)\right\|_{3/2}=0,\ \lim_{t\to T-0}\left\|u^i(t)\right\|_{1/2}=0,$$

then $u(t)$ will be called a regular solution of problem (1), (2).

**Definition 3**. If for any $f(t)\in L_2((0,T);H)$ there is a regular solution $u(t)\in W_2^2(R_+;H)$ problem (1), (2) and the following estimates hold

$$\|u\|_{W_2^2((0,T);H)}\le const\,\|f\|_{L_2((0,T);H)},$$

then we say that problem (1), (2) is regularly solvable.

## 4. THE MAIN RESULTS

First, we will investigate the regular solvability of the problem

$$-\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t)=f(t) \qquad (3)$$

$$u(0)=Su'(0), u(T)=0. \qquad (4)$$

Denote by

$$P_0\left(\frac{d}{dt}\right)=-\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t),\ t\in(0,T), u\in \overset{0}{W}{}^2_{2,S}((0,T);H),$$

$$P_1\left(\frac{d}{dt}\right)=A_1(t)\frac{du(t)}{dt}+A_2(t)u(t),\ t\in(0,T), u\in \overset{0}{W}{}^2_{2,S}((0,T);H),$$

$$Pu=P_0u+P_1u,\ u\in \overset{0}{W}{}^2_{2,S}((0,T);H).$$

First, let's prove some assertions

**Lemma 1.** Let conditions 1), 2) and 4) be satisfied, $\operatorname{Re}AS\ge 0$ in $H_{1/2}$. Then the equation $P_0u=0$ has only zero solution.

**Proof**. Let $u\in \overset{0}{W}{}^2_{2,S}((0,T);H)$. Then from equation (3) we obtain that

$$\rho^{-1/2}\frac{d^2u(t)}{dt^2}+\rho^{1/2}(t)A^2u(t)=\rho^{-1/2}f(t),\ t\in(0,T).$$

Hence we have:

$$\left\|\rho^{-1/2}\frac{d^2u(t)}{dt^2}+\rho^{1/2}(t)A^2u(t)\right\|^2_{L_2((0,T);H)}=\left\|\rho^{-1/2}P_0u\right\|^2_{L_2((0,T);H)}.$$

Thus,

$$=2\operatorname{Re}\left(u'(0),ASu'(0)\right)+2\operatorname{Re}\left\|A\tfrac{du}{dt}\right\|^2_{L_2((0,T);H)}=\left\|\rho^{-1/2}P_0u\right\|^2_{L_2((0,T);H)}+\left\|\rho^{-1/2}(t)\frac{d^2u}{dt^2}\right\|^2_{L_2((0,T);H)}-$$

$$-2\operatorname{Re}\left(\rho^{-1/2}(t)\frac{d^2u}{dt^2},\rho^{1/2}(t)A^2u(t)\right)_{L_2((0,T);H)}. \quad (5)$$

Integrating by parts, we get

$$-2\operatorname{Re}\left(\rho^{-1/2}(t)\frac{d^2u}{dt^2},\rho^{1/2}(t)A^2u(t)\right)_{L_2((0,T);H)}=\left(\frac{d^2u}{dt^2},A^2u(t)\right)_{L_2((0,T);H)}=$$

$$=-2\operatorname{Re}\int_0^T\left(\frac{d^2u}{dt^2},A^2u(t)\right)dt=-2\operatorname{Re}\left(A^{1/2}u(t),A^{3/2}u(t)\right)\Big|_0^T+2\operatorname{Re}\int_0^T\left(\frac{d^2u}{dt^2},A^2u(t)\right)dt=$$

$$=2\operatorname{Re}\left[\left(A^{1/2}u'(T),A^{3/2}u(T)\right)-\left(A^{1/2}u'(0),A^{3/2}u(0)\right)\right]+2\operatorname{Re}\left\|A\tfrac{du}{dt}\right\|^2_{L_2((0,T);H)}=$$

$$=2\operatorname{Re}\left(A^{1/2}u'(0),A^{3/2}Su'(0)\right)+2\operatorname{Re}\left\|A\tfrac{du}{dt}\right\|^2_{L_2((0,T);H)}=$$

$$=2\operatorname{Re}\left(u'(0),ASu'(0)\right)+2\operatorname{Re}\left\|A\tfrac{du}{dt}\right\|^2_{L_2((0,T);H)}.$$

Since, according to the hypothesis of the theorem $\operatorname{Re}\left(u'(0),ASu'(0)\right)\geq 0$, then we get that

$$2\operatorname{Re}\left(\rho^{-1/2}(t)\frac{d^2u}{dt^2},\rho^{1/2}(t)A^2u(t)\right)_{L_2((0,T);H)}\geq 2\operatorname{Re}\left\|A\tfrac{du}{dt}\right\|^2_{L_2((0,T);H)}. \quad (6)$$

Taking into account inequality (6) in (5), we obtain

$$\left\|\rho^{-1/2}P_0u\right\|^2_{L_2((0,T);H)}\geq\left\|\rho^{-1/2}\frac{d^2u(t)}{dt^2}\right\|^2_{L_2((0,T);H)}+\left\|\rho^{-1/2}A^2u\right\|^2_{L_2((0,T);H)}+\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}. \quad (7)$$

Therefore, if $P_0u=0$, then it follows from inequality (7)

$$\left\|\rho^{-1/2}\frac{d^2u(t)}{dt^2}\right\|^2_{L_2((0,T);H)}+\left\|\rho^{-1/2}A^2u\right\|^2_{L_2((0,T);H)}+\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}=0.$$

Therefore $A^2u=0$, i.e. $u=0$.

Let us now prove the regular solvability theorem for problem (3), (4).

**Theorem 1**. *Let conditions 1), 2), 4)* be satisfied, $\operatorname{Re}AS\geq 0$ *в* $H$. *Then problem (3), (4) is regularly solvable.*

**Proof**. We write problem (3), (4) in the form of the equation

$$P_0u=f,\ f(t)\in L_2((0,T);H),\ u\in\overset{0}{W}{}^2_{2,S}((0,T);H).$$

It follows from Lemma 1 that $KerP_0=\{0\}$. Now let us prove that the image of the operator $P_0:\overset{0}{W}{}^2_{2,S}((0,T);H)\to L_2((0,T);H)$ coincides with the space $f(t)\in L_2((0,T);H)$, i.e. $P_0u=f$ has a solution for any $f(t)\in L_2((0,T);H)$. Let

$$f_1(t)=\begin{cases}f(t),\ t\in(0,T)\\ 0\quad ,t\in R/(0,T)\end{cases}.$$

Obviously, $f_1(t)\in L_2(R;H)$, $\|f_1\|_{L_2(R:H)}=\|f\|_{L_2((0,T);H)}$. Denote by $\hat{f}_1(\xi)$ Fourier transform of a vector -

functions $f_1(t)$, i.e.

$$\hat{f}_1(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(t)e^{-i\xi t}dt.$$

Then we define the vector - functions

$$u_\alpha(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\xi^2E+\alpha^2A^2\right)^{-1}\hat{f}_1(t)e^{-i\xi t}dt,$$

$$u_\beta(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\xi^2E+\beta^2A^2\right)^{-1}\hat{f}_1(t)e^{i\xi t}dt,$$

at $t\in R=(-\infty,\infty)$. Show that $u_\alpha(t),\ u_\beta(t)\in W_2^2(R:H)$ Indeed, by the Plansherel theorem

$$\left\|u_\alpha(t)\right\|^2_{W_2^2(R;H)}=\left\|\frac{d^2u_\alpha}{dt^2}\right\|^2_{L_2((0,T);H)}+\left\|A^2u_\alpha\right\|^2_{L_2((0,T);H)}=\left\|A^2\hat{u}_\alpha(\xi)\right\|^2_{L_2((0,T);H)}+\left\|\xi^2\hat{u}_\alpha(\xi)\right\|^2_{L_2((0,T);H)}.$$

Taking into account that $\hat{u}_\alpha(t)=\left(\xi^2E+\alpha^2A^2\right)^{-1}\hat{f}_1(\xi)$, obtain that

$$\left\|u_\alpha(t)\right\|^2_{W_2^2(R;H)}=\left\|\left(\xi^2E+\alpha^2A^2\right)^{-1}\hat{f}_1(\xi)\right\|^2_{L_2(R;H)}+\left\|\left(A^2E+\alpha^2A^2\right)^{-1}\hat{f}_1(\xi)\right\|^2_{L_2(R;H)}\le$$

$$\le\sup_t\left\|\left(\xi^2E+\alpha^2A^2\right)^{-1}\right\|^2_{L_2(R;H)}\left\|\hat{f}_1(\xi)\right\|^2_{L_2(R;H)}+\sup_t\left\|\left(A^2E+\alpha^2A^2\right)^{-1}\right\|^2_{L_2(R;H)}\left\|\hat{f}_1(\xi)\right\|^2_{L_2(R;H)}=$$

$$\left(\sup_t\left\|\left(\xi^2E+\alpha^2A^2\right)^{-1}\right\|^2_{L_2(R;H)}+\sup_t\left\|\left(A^2E+\alpha^2A^2\right)^{-1}\right\|^2_{L_2(R;H)}\right)\left\|\hat{f}_1(\xi)\right\|^2_{L_2(R;H)}.$$

Next, using the spectral decomposition of the operator $A$ for any $\xi\in R$ obtain

$$\left\|\left(\xi^2E+\alpha^2A^2\right)^{-1}\hat{f}_1(\xi)\right\|=\sup_{\sigma\in\sigma(A)}\left\|\xi^2\left(\xi^2E+\alpha^2A^2\right)^{-1}\right\|\le 1 \text{ и } \left\|A^2\left(A^2E+\alpha^2A^2\right)^{-1}\right\|=$$

$$=\sup_{\sigma\in\sigma(A)}\left\|\sigma^2\left(\xi^2E+\alpha^2\sigma^2\right)^{-1}\right\|\le\frac{1}{\alpha^2}.$$

Then $\left\|u_\alpha(t)\right\|^2_{W_2^2(R;H)}\le\left(1+\frac{1}{\alpha^4}\right)\left\|f\right\|^2_{L_2((0,T);H)}$. Hence $u_\alpha(t)\in W_2^2(R:H)$. Similarly, it is proved that $\left\|u_\beta(t)\right\|^2_{W_2^2(R;H)}\le\left(1+\frac{1}{\beta^4}\right)\left\|f\right\|^2_{L_2((0,T);H)}$, i.e. $u_\beta(t)\in W_2^2(R:H)$. Then it is obvious that

$$\xi_\alpha=\begin{cases}u_\alpha(t), & t\in(0,T)\\ 0, & t\in R/(0,T)\end{cases},$$

$$\xi_\beta=\begin{cases}u_\beta(t), & t\in(0,T)\\ 0, & t\in R/(0,T)\end{cases}$$

belong to the space $W_2^2((0,T):H)$ and according to the trace theorem

$$\xi_\alpha^{(j)}(0)\in H_{2-j-1/2},\ \xi_\alpha^{(j)}\in H_{2-j-1/2},$$

$$\xi_\beta^{(j)}(0)\in H_{2-j-1/2},\ \xi_\beta^{(j)}\in H_{2-j-1/2},\ j=0,1.$$

On the other side $\xi_\alpha(t)$ satisfies the equation

$$-\frac{d^2u(t)}{dt^2}+\alpha^2A^2u(t)=f(t)$$

but $\xi_\beta(t)$ satisfies the equation

$$-\frac{d^2u(t)}{dt^2}+\beta^2A^2u(t)=f(t)$$

in $[0,T]$ almost everywhere because $u_\alpha(t)$, $u_\beta(t)$ satisfies the equation

$$-\frac{d^2u(t)}{dt^2}+\alpha^2A^2u(t)=f_1(t),$$
$$-\frac{d^2u(t)}{dt^2}+\beta^2A^2u(t)=f_1(t)$$

in $R=(-\infty,\infty)$ almost everywhere. Now consider the vector function

$$u(t)=\begin{cases}\xi_\alpha(t)+e^{-\alpha At}\varphi_1+e^{-\alpha(t-t_0A)}\varphi_2\ ,t\in(0,t_0)\\ \xi_\beta(t)+e^{-\beta At}\varphi_3+e^{-\beta(T-t)A}\varphi_4\ ,t\in(t_0,T)\end{cases}$$

where are the vectors $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ so for unknown vectors that belong to the definition. Obviously, if we show that $\varphi_1, \varphi_2, \varphi_3, \varphi_4\in H_{3/2}$ and $u_\alpha(t)\in \overset{0}{W}{}^2_{2,S}((0,T);H)$, we get that $u(t)\in \overset{0}{W}{}^2_{2,S}((0,T);H)$, $P_0u=P_0\left(d/dt\right)u(t)==f(t),\ t\in[0,T]$. Then from the condition $u(t)\in \overset{0}{W}{}^2_{2,S}((0,T);H)$ follows that

$$\begin{cases}\xi_\alpha(0)+\varphi_1+e^{-\alpha t_0A}\varphi_2=S\left(\xi'(0)-\alpha A\varphi_1+\alpha Ae^{-\alpha t_0A}\varphi_2\right)\\ \xi_\beta(T)+e^{-\beta(T-t_0)At}\varphi_3+\varphi_4=0\\ \xi_\alpha(t_0)+e^{-\alpha t_0A}\varphi_1+\varphi_2=\xi_\beta(t)\varphi_3+e^{\beta(t_0-T)A}\varphi_4\\ \xi_\alpha(t_0)-\alpha Ae^{-\alpha t_0A}\varphi_1+\alpha\varphi_2=\xi_\beta(t_0)-\beta A\varphi_.+\beta Ae^{\beta(t_0-T)A}\varphi_4\end{cases}$$

From the second equation we obtain that $\varphi_4=-e^{\beta(t_0-T)A}\varphi_3-\xi_\beta(T)$. Hence we get

$$\begin{cases}\varphi_1+\alpha SA\varphi_1+e^{-\alpha t_0A}\varphi_2-\alpha Ae^{-\alpha t_0A}\varphi_2=S\left(\xi'(0)\right)-\xi_\alpha(0)\\ e^{-\beta(T-t_0)At}\varphi_3+\varphi_4=-\xi_\beta(T)\\ e^{-\alpha t_0A}\varphi_1+\varphi_2-\varphi_3-e^{\beta(t-T)A}=\xi_\beta(t_0)-\xi_\alpha(t_0)\\ -\alpha Ae^{-\alpha t_0A}\varphi_1+\alpha\varphi_2+\beta A+\beta Ae^{\beta(t_0-T)A}\varphi_4=\xi'_\beta(t_0)-\xi_\alpha(t_0)\end{cases}.$$

It's obvious that $A(\xi'_\alpha(0))-\xi_\alpha(0)\in H_{3/2}$, $S(\xi'_\alpha(0))-\xi_\alpha(0)\in H_{3/2}$, $-\xi'_\beta(T_0)\in H_{3/2}$, $\xi_\beta(t_0)-\xi_\alpha(t_0)\in H_{3/2}$, $\xi'_\beta(t_0)-\xi'_\alpha(t_0)\in H_{3/2}$. It follows from the second equation that $\varphi_4=-e^{\beta(t_0-T)A}\varphi_3-\xi_\beta(H)$. Considering this expression in the third and fourth equations, we obtain

$$e^{\alpha t_0A}\varphi_1+\varphi_2-\left(E-e^{2\beta(t_0-T)A}\right)\varphi_3=\xi_\beta(t_0)-\xi_\varepsilon(t_0)-e^{\beta(t_0-T)}\xi_\alpha(T)-\alpha e^{-\alpha t_0A}\varphi_1+\alpha\varphi_2+$$
$$+\beta\left(E+e^{2\beta(t_0-T)A}\right)\varphi_3=A^{-1}\left(\xi'_\beta(t_0)-\xi'_\alpha(t_0)\right).$$

Thus, in order to define $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ we obtain the following system of equations

$$\begin{cases}(E+\alpha SA)\varphi_1+(E+\alpha SA\varphi_1)e^{-\alpha t_0A}\varphi_2=\psi_1\\ \varphi_1-e^{-\beta(T-t_0)At}\varphi_3=\psi_2\\ e^{-\alpha t_0A}\varphi_1+\varphi_2-\left(E-e^{2\beta(t-T)A}\varphi_3\right)=\psi_3\\ -\alpha e^{-\alpha t_0A}\varphi_1-\varphi_2-\frac{\beta}{2}\left(E+Ae^{\beta(t_0-T)A}\right)\varphi_3=\psi_4\end{cases}\qquad(8)$$

where $\psi_1=\xi_\beta(t_0)-\xi_\alpha(t_0)-e^{2\beta(t_0-T)A}\xi_\alpha(T)$, $\psi_2=-\xi_\beta(T)$, $\psi_3=\xi_\beta(t_0)-\xi_\alpha(t_0)$, $\psi_4=\xi_\beta(t_0)-\xi_\alpha(t_0)$ Hence it follows that $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4\in H_{3/2}$, moreover, all vectors $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ known. It follows from the third and fourth equations of system (8) that

$$2\varphi_2=\left(\left(1-\frac{\beta}{\alpha}\right)E-\left(1+\frac{\beta}{\alpha}\right)e^{2\beta(t_0-T)}\right)\varphi_3+(\psi_3+\psi_4)\qquad(9)$$

and

$$2e^{-2t_0A}\varphi_1 = \left(\left(1+\frac{\beta}{\alpha}\right)E + \left(\frac{\beta}{\alpha}-1\right)e^{2\beta(t_0-T)}\right)\varphi_3 + (\psi_3+\psi_4). \tag{10}$$

Thus, from (10) it follows that

$$\varphi_3 = \left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)}\right)^{-1}\alpha\left(2e^{-\alpha t_0 A}\varphi_1 - \psi_3 - \psi_4\right). \tag{11}$$

For the correctness of the definition, we show that the operator $\left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)}\right)^{-1}$ exists and is limited. From the spectral expansion of the operator $A$ follows that

$$\left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)}\right)^{-1} = \int_{\mu_0}^{\infty}\frac{1}{(\alpha+\beta)+(\beta-\alpha)e^{2\beta(t_0-T)\sigma}}\,dE_\omega\,,$$

at $\sigma \in \sigma(A)(\sigma \ge \mu_0 > 0)$

$$\left|\frac{1}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0-T)\sigma}}\right| \le \frac{1}{\alpha+\beta}\left|\frac{1}{1+\dfrac{(\beta-\alpha)}{\beta+\alpha}e^{2\beta(t_0-T)\sigma}}\right| \le \frac{1}{\alpha+\beta}\frac{1}{1-\left|\dfrac{\beta-\alpha}{\beta+\alpha}\right|e^{2\beta(t_0-T)\sigma}} <$$

$$< \frac{1}{\alpha+\beta}\frac{1}{1-\left|\dfrac{\beta-\alpha}{\beta+\alpha}\right|} = \frac{1}{\alpha+\beta-|\beta-\alpha|}.$$

Thus, taking into account (11) in (10), we obtain

$$2\varphi_2 = \left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)}\right)\left[\left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)A}\right)^{-1}\left(2e^{-\alpha t_0 A}\varphi_1 - \psi_3 - \psi_4\right)\right]$$

or

$$2\varphi_2 = \left((\alpha-\beta)E - (\beta+\alpha)e^{2\beta(t_0-T)}\right)\left[\left((\alpha+\beta) + (\beta-\alpha)e^{2\beta(t_0-T)A}\right)\right]^{-1}e^{-\alpha t_0 A} - \frac{1}{2}(\psi_3-\psi_4).$$

Then it follows from the first equation of system (8) that

$$(E+\alpha SA)\varphi_1 + (E-\alpha SA)e^{-\alpha(t_0-T)A}\left((\alpha-\beta)E - (\beta+\alpha)e^{2\beta(t_0-T)}\right),$$

$$\left((\alpha-\beta)E - (\beta+\alpha)e^{2\beta(t_0-T)}\right)^{-1}\left(e^{-\alpha t_0 A}\varphi_1 - \frac{1}{2}(\psi_3+\psi_4)\right).$$

Next, we denote $\chi_1 = A\varphi_1 \in H_{1/2}$ and acting on both sides of the last equation we get $(E+\alpha SA)\chi_1 + (E-\alpha SA)Q\chi_1 = \chi_1$ , where

$$Q = e^{-\alpha t_0 A},$$

$$Q_1 = \left((\alpha-\beta)E - (\beta+\alpha)e^{2\beta(t_0-T)}\right)\left[\left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)A}\right)\right]^{-1},$$

$$\chi_1 = A\psi_1 + \frac{1}{2}(E-\alpha SA)\,e^{-\alpha t_0 A}Q_1(\psi_3+\psi_4).$$

It's obvious that $\chi_1 \in H_{1/2}$. Thus, in $H_{1/2}$ has the following equation

$$(E+Q)\chi_1 + \alpha AS(E-Q)\chi_1 = \chi_1 \tag{12}$$

Denote by

$$F = (E+Q) + \alpha AS(E-Q) : H_{1/2} \to H_{1/2} \tag{13}$$

and show that the operator $F$ turn to $H_{1/2}$. To do this, we estimate the norm of the operator $Q$. Now let's show that $\|Q\| < 1$. Let

$$Q_1 = \left((\alpha-\beta)E - (\beta+\alpha)e^{2\beta(t_0-T)}\right)\left[\left((\alpha+\beta)E + (\beta-\alpha)e^{2\beta(t_0-T)A}\right)\right]^{-1}.$$

It's obvious that

$$\|Q_1\| = \sup_{\sigma\in\sigma(A)}\left|\frac{(\alpha-\beta)-(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0T)A}}\right|.$$

Let us show that $\|Q\|<1$. Let 1) $\alpha<\beta$. Then

$$\sup_{\sigma\in\sigma(A)}\left|\frac{(\alpha-\beta)-(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0T)A}}\right|=f(\sigma).$$

It is obvious that at $\sigma\geq\mu_0>0$

$$f^{'}(\sigma)=2\beta(t_0-T)e^{2\beta(t_0-T)\sigma}4\alpha\beta=8\alpha\beta^2(t_0-T)e^{2\beta(t_0-T)\sigma}<0.$$

Thus, $f(\sigma)$ monotonically decreasing function, so $\sup_{\sigma\in\sigma(A)}\left|\frac{(\alpha-\beta)-(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0T)A}}\right|<1$

because from the inequality $\beta-\alpha+(\beta+\alpha)e^{2\beta(t_0-T)\sigma}<\beta+\alpha+(\beta-\alpha)e^{2\beta(t_0-T)\sigma}$. Follows that $e^{2\beta(t_0-T)\sigma}<2\alpha$. Because $t_0-T<0,\alpha>0$, then this inequality is true.

2) Let $\alpha>\beta$, $\alpha>\beta$ and $\beta-\alpha>(\beta+\alpha)e^{2\beta(t_0-T)\sigma}$, $\sigma>\frac{1}{2\beta}(t_0-T)^{-1}\ln\frac{\alpha-\beta}{\alpha+\beta}$.

$$\left|\frac{(\alpha-\beta)-(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0T)A}}\right|=\frac{(\beta-\alpha)+(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\alpha-\beta)e^{2\beta(t_0-T)\sigma}}.$$

Because $f^{'}(\sigma)=2\beta(t_0-T)e^{2\beta(t_0-T)\sigma}4\alpha\beta=8\alpha\beta(t_0-T)e^{2\beta(t_0-T)\sigma}>0$, i.e. $f(\sigma)$ increases monotonically

$$\sup_{\sigma\in\sigma(A)}\left|\frac{(\alpha-\beta)-(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0T)A}}\right|=\frac{\alpha-\beta}{\alpha+\beta}<1.$$

Now $\alpha>\beta$,

$$\beta-\alpha>(\beta+\alpha)e^{2\beta(t_0-T)\sigma},\ \text{i.e. } \sigma>\frac{1}{2\beta}(t_0-T)^{-1}\ln\frac{\alpha-\beta}{\alpha+\beta}$$

$$\left|\frac{(\alpha-\beta)-(\alpha+\beta)e^{2\beta(t_0-T)\sigma}}{\alpha+\beta+(\beta-\alpha)e^{2\beta(t_0T)A}}\right|=\frac{(\alpha+\beta)e^{2\beta(t_0-T)\sigma}-(\alpha-\beta)}{\alpha+\beta+(\alpha-\beta)e^{2\beta(t_0-T)\sigma}}=f(\sigma).$$

Then

$$f^{'}(\sigma)=2\beta(t_0-T)e^{2\beta(t_0-T)\sigma}4\alpha\beta=8\alpha\beta(t_0-T)e^{2\beta(t_0-T)\sigma}(t_0-T),$$

i.e. function $f(\sigma)$ for $\mu<\sigma<\frac{1}{2\beta}(t_0-T)^{-1}\ln\frac{\alpha-\beta}{\alpha+\beta}$ is decreasing and $\sup\|f^{'}(\sigma)\|<1$. Hence $\|Q\chi\|=\|e^{-\alpha t_0A}Q_1\chi\|<e^{-\alpha t_0A}\|\chi\|\leq\theta\|\chi\|$, $\theta<1$. Then for the solution of equation (12) we show that $\|F\chi\|_{1/2}\geq C\|\chi\|_{1/2}$. Since at $\chi\in H_{1/2}$ $F\chi=(E+Q)\chi+\alpha AS(E-Q)\chi$, then we multiply by $H_{1/2}$ both parts of this equality are scalar on $(E-Q)\chi$ в $H_{1/2}$ we have

$$(F\chi,(E-Q)\chi)=((E+Q)\chi,(E-Q)\chi)+(\alpha AS(E-Q)\chi,(E-Q)\chi).$$

Then $\operatorname{Re}(F\chi,(E-Q)\chi)=\operatorname{Re}((E^2-Q^2)\chi,\ \chi)+\operatorname{Re}(\alpha AS(E-Q)\chi,(E-Q)\chi)=\|\chi\|^2-\|Q\chi\|^2+$ $+\alpha\operatorname{Re}(\alpha AS(E-Q)\chi,(E-Q)\chi)$. Because $\alpha\operatorname{Re}(\alpha AS(E-Q)\chi,(E-Q)\chi)\geq0$, $\alpha>0$, that

$$\operatorname{Re}(F\chi,(E-Q)\chi)\geq\|\chi\|^2-\|Q\chi\|^2=(1-\theta)\|\chi\|^2.$$

Hence

$$(1-\theta)\|\chi\|^2\leq\operatorname{Re}(F\chi,(E-Q)\chi)\leq\|F\chi\|\|(E-Q)\chi\|\leq\|F\chi(1+\theta)\|\|\chi\|.$$

Hence we have that $\|F\chi\|\geq\frac{1-\theta}{1+\theta}\|\chi\|=C\|\chi\|$. If $K=AS$, $\operatorname{Re}K\geq0$

$$F^*=(E+Q)+\alpha(E-Q)K \text{ and } F^*\varphi=(E+Q)+\alpha(E-Q)K\chi$$
$$F^*(E-Q)=(E^2-Q^2)+\alpha(E-Q)K(E-Q)$$

and

$$F^*(E-Q)\chi,\ \chi=(E^2-Q^2)(\chi,\chi)+\alpha(K(E-Q)\chi,(E-Q)\chi).$$

$\operatorname{Re}(F^*(E-Q)\chi,\ \chi)\geq(1-\theta^2)\|\chi\|^2$. If $(E-Q)\chi=y$, then

$$(F^*y,(E-Q)^{-1}y)\geq(1-\theta^2)\|(E-Q)^{-1}y\|^2 \text{ or } \|(1-\theta^2)(E-Q)^{-1}y\|\leq\|F^*y\|.$$

Because $\|(1-\theta^2)(E-Q)^{-1}y\|\geq(1-\theta)y$, then $\|F^*y\|\geq C_1\|y\|^2, C_1>0$. Then $F$ reversible, so we can solve equation (12): $\chi_1=F^{-1}\chi_1$. Because $\chi_1=A\varphi_1, \varphi_1=A^{-1}\chi_1\in H_{3/2}$. Then, obviously, we can define $\varphi_2,\varphi_3$, $\varphi_4\in H_{3/2}$. Thus, the vector function $u(t)\in W^2_{2,s}((0,T);H)$. On the other side

$$\|P_0u\|^2_{L_2((0,T);H)}=\left\|-\frac{d^2u}{dt^2}+\rho(t)A^2u\right\|^2_{L_2((0,T);H)}\leq 2\left(\left\|-\frac{du}{dt}\right\|^2_{L_2((0,T);H)}+\max_t\rho(t)\right)\|A^2u\|^2_{L_2((0,T);H)}$$

we get that the operator $P_0$ continuous out $\overset{0}{W}{}^2_{2,s}((0,T);H)$ on the $L_2((0,T);H)$. Then the Banach theorem on the inverse operator implies the assertion of the theorem. The theorem is proved.

**Lemma 2**. *For any $u(t)\in W^2_{2,S}((0,T);H)$ the following inequalities hold*

$$\left\|A^2\frac{du}{dt}\right\|_{L_{21}((0,T);H)}\leq\frac{1}{\min(\alpha^2+\beta^2)}\|P_0u\|_{L_2((0,T);H)},\qquad (14)$$

$$\left\|A\frac{du}{dt}\right\|_{L_{21}((0,T);H)}\leq\frac{1}{\min(\alpha,\beta)}\|P_0u\|_{L_2((0,T);H)}.\qquad (15)$$

**Proof**. It follows from the proof of Lemma 1 that

$$\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2((0,T);H)}\geq\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2((0,T);H)}+\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)}+2\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}.\qquad (16)$$

On the other hand at $u(t)\in\overset{0}{W}{}^2_{2,S}((0,T);H)$

$$\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}=\int_0^T\left(A\frac{du}{dt},A\frac{du}{dt}\right)=\left(A^{3/2}u(t),A^{1/2}u'(t)\right)\Big|_0^T--\operatorname{Re}\left(A^{1/2}ASu'(0),\ A^{1/2}u'(0)\right)+$$
$$+\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)}+\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2((0,T);H)}.$$

Hence we get that

$$\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)}+\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2((0,T);H)}\geq 2\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}+2\operatorname{Re}\left(ASu'(\ ),u'(0)\right)_{1/2}.$$

Because $2\operatorname{Re}\left(ASu'(\ ),u'(0)\right)_{1/2}\geq 0$, then

$$\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)}+\left\|\rho^{-1/2}\frac{d^2u}{dt^2}\right\|^2_{L_2((0,T);H)}\geq 2\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}.\qquad (17)$$

Taking into account inequality (17) in (16), we obtain:

$$\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)}\geq 4\left\|A\frac{du}{dt}\right\|^2_{L_2((0,T);H)}$$

or

$$\left\|A\frac{du}{dt}\right\|_{L_2((0,T);H)}\leq\frac{1}{2}\left\|\rho^{-1/2}P_0u\right\|_{L_2((0,T);H)}=\frac{1}{2\min(\alpha,\beta)}\left\|\rho^{-1/2}P_0u\right\|_{L_2((0,T);H)},$$

i.e. the correctness of inequality (15) is proved. On the other hand, it follows from (16) that

$$\left\|\rho^{1/2}A^2u\right\|^2_{L_2((0,T);H)} \le \left\|\rho^{-1/2}P_0u\right\|^2_{L_2((0,T);H)}, \text{ i.e. } \left\|\rho^{1/2}A^2u\right\|_{L_2((0,R);H)} \le \left\|\rho^{-1/2}P_0u\right\|_{L_2((0,T);H)}.$$

Hence we have

$$\left\|A^2u\right\|_{L_2((0,T);H)} = \left\|\rho^{-1/2}\rho^{1/2}u\ A^2u\right\|_{L_2((0,T);H)} \le \max_t \rho^{-1/2}(t)\left\|\rho^{1/2}u\ A^2u\right\|_{L_2((0,T);H)} \le$$

$$\le \frac{1}{\min(\alpha^2,\beta^2)}\left\|P_0u\right\|_{L_2((0,T);H)}.$$

Inequality (14) is also proved.

**Lemma 3**. *Let conditions 1)-3) be satisfied. Then*

$$\left\|P_1u\right\|_{L_2((0,T);H)} = \left\|A_1(t)\frac{du}{dt} + A_2(t)u\right\|_{L_2((0,T);H)} \le \left\|A_1(t)\frac{du}{dt}\right\|_{L_2((0,T);H)} + \left\|A_2(t)u\right\|_{L_2((0,T);H)} \le$$

$$\le \sup_{t\in(0,T)}\left\|A_1(t)A^{-1}\right\|_{L_2((0,T);H)}\left\|A\frac{du}{dt}\right\|_{L_2((0,T);H)} + \frac{1}{\min(\alpha^2,\beta^2)}\sup_{t\in(0,T)}\left\|A_2(t)A^{-2}\right\|_{L_2((0,T);H)} \times$$

$$\times\left\|A_2\,u\right\|_{L_2((0,T);H)}. \quad (18)$$

It follows from the intermediate derivatives theorem that

$$\left\|A\frac{du}{dt}\right\|_{L_2((0,T);H)} \le C_1\left\|u\right\|_{W_2^2(0,T);H} \text{ and } \left\|A_2\,u\right\|_{L_2((0,T);H)} \quad \left\|A\frac{du}{dt}\right\|_{L_2((0,T);H)} \le C_2\left\|u\right\|_{W_2^2(0,T);H}.$$

Considering this, in (18) it follows that

$$\left\|P_1u\right\|_{L_2((0,T);H)} \le const\ \left\|u\right\|_{W_2^2(0,T);H}.$$

The lemma is proved.

It follows from this lemma and theorem 1 that the operator

$$P = P_0 + P_1 : \overset{0}{W}{}^2_{2,s}((0,T);H) \to L_2((0,T);H)$$

continuous.

Now let's prove the main theorem.

**Theorem 2**. *Let conditions 1)-4),* $\operatorname{Re} AS \ge 0$ *in* $H_{1/2}$ *and there is an inequality*

$$\delta = \frac{1}{2\min(\alpha,\beta)}\sup_{t\in(0,T)}\left\|A_1(t)A^{-1}\right\|_{L_2((0,T);H)} + \frac{1}{\min(\alpha^2,\beta^2)}\sup_{t\in(0,T)}\left\|A_2(t)A^{-12}\right\|_{L_2((0,T);H)}. \quad (19)$$

*Then problem (1), (2) is regularly solvable.*

**Proof.** Let us write problem (1), (2) as an equation $Pu = P_0u + P_1u$, which $u \in \overset{0}{W}{}^2_{2,s}((0,T);H)$, $f \in L_2((0,T);H)$. By theorem 1, the operator

$$P_0 : \overset{0}{W}{}^2_{2,s}((0,T);H) \to L_2((0,T);H)$$

mutually unique. Then for any $\omega \in L_2((0,T);H)$ exists $u \in \overset{0}{W}{}^2_{2,s}((0,T);H)$, such that

$$P_0u = \omega.$$

Then relatively $\omega$ we obtain the equation $\omega + P_1P_0^{-1} = f$ in the space $L_2((0,T);H)$. Then for any $\omega \in L_2((0,T);H)$ have

$$\left\|P_1P_0^{-1}\omega\right\|_{L_2((0,T);H)} = \left\|A_1(t)\frac{du}{dt}\right\|_{L_2((0,T);H)} + \left\|A_2(t)\right\|_{L_2((0,T);H)} = \left\|A_1(t)A^{-1}A\frac{du}{dt}\right\|_{L_2((0,T);H)} +$$

$$+\left\|A_2(t)A^{-2}A^2\right\|_{L_2((0,T);H)} \le \sup_t\left\|A_1(t)A^{-1}A\frac{du}{dt}\right\|_{L_2((0,T);H)} + \sup_t\left\|A_2(t)A^{-2}A^2\right\|_{L_2((0,T);H)}. \quad (20)$$

Taking into account inequalities (14) and (15) in (20), we obtain

$$\left\|P_1 P_0^{-1}\omega\right\|_{L_2((0,T);H)} = \frac{1}{2\min(\alpha,\beta)}\sup_{t\in(0,T)}\left\|A_1(t)A^{-1}\right\|_{L_2((0,T);H)} +$$

$$+\frac{1}{\min(\alpha^2,\beta^2)}\sup_{t\in(0,T)}\left\|A_2(t)A^{-12}\right\|_{L_2((0,T);H)} = \delta\|\omega\|_{L_2((0,T);H)}.$$

Therefore, the operator $E+P_1P_0^{-1}$ turn in $L_2((0,T);H)$. Then $\omega=\left(E+P_1P_0^{-1}\right)^{-1}f$. Hence we have that $u=P_0^{-1}\left(E+P_1P_0^{-1}\right)^{-1}f$ and

$$C_2\|u\|_{W_2^2(0,T);H} \le \left\|P_0^{-1}\right\|_{L_2((0,T);H)\to \overset{0}{W}{}_{2,s}^2((0,T);H)}\frac{1}{1-\delta}\|f\|_{L_2(|0,T|;H)} = const\ \|f\|_{L_2((0,T);H)}.$$

The theorem is proved.

**Corollary**. *Let conditions 1)–3) be satisfied and inequality (19) hold. Then the problem is*

$$-\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t)+A_1(t)\frac{du(t)}{dt}+A_2(t)u(t)=f(t) \qquad (21)$$

$$u(0)=0,u(T)=0, \qquad (22)$$

*regularly solvable.*

The proof follows from theorem 2 when $S=0$. Note that if problem (21), (22) is regularly solvable, we can prove a more general statement.

**Theorem 3**. *Let conditions 1), 3) and* $\rho(t)$ *measurable scalar function such that* $0<\alpha\le\rho(t)\le\beta$ *and there is an inequality*

$$\delta=\frac{1}{2\sqrt{\alpha}}\sup_{t\in(0,T)}\left\|A_1(t)A^{-1}\right\|+\frac{1}{\alpha}\sup_{t\in(0,t)}\left\|A_2(t)A^{-2}\right\|<1$$

*Then problem (21), (22) is regularly solvable.*

**Proof**. Here we also denote by

$$P_0u=P_0\left(\frac{d}{dt}\right)u=-\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t),\ P_1u=P_1\left(\frac{d}{dt}\right)=A_1(t)\frac{du(t)}{dt}+A_2(t)u(t),$$

$$u(t)\in W_2^2((0,T);(0,0)).$$

It's obvious that $P=P_0+P_1:W_2^2((0,T);(0,0))\to L_2((0,T);H)$ continuous. From lemma 1 for $S=0$ follows that $Ker\,P_0=\{0\}$. Let us prove that the operator $P_0$ displays space $W_2^2((0,T);(0,0))$ on $L_2((0,T);H)$. If we consider the operator $L_0$ in the space $L_2((0,T);H)$ generated by an operator-differential expression

$$P_0\left(\frac{d}{dt}\right)u=-\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t),$$

with domain of definition $D(L_0)=\{u:u\in W_2^2((0,T);(0,0))\}$, then we get that the operator $L_0$ positive-definite self-adjoint operator in space $L_2((0,T);H)$ with scope $W_2^2((0,T);(0,0))$ $(u(0)=0,u(T)=0)$. Really

$$(L_0u,v)_{L((0,T);H)}=(u,L_0v)_{L_2((0,T);H)},$$

and

$$(L_0u,u)_{L((0,T);H)}=\int_0^T\left(-\frac{d^2u}{dt^2},u\right)dt+\int_0^T\left(\rho(t)A^2u,u\right)dt\ge\alpha\|Au\|^2_{L_2((0,T);H)}\ge\alpha\mu_0^2(u,u)_{L_2((0,T);H)}=$$

$$=\alpha\mu_0^2\|u\|^2_{L_2((0,T);H)},$$

i.e. $L_0=\alpha\mu_0^2E$ in $L_2((0,T);H)$. Then the operator $L_0$ turn into $L_2((0,T);H)$, $L_0^{-1}$ bounded in space $L_2((0,T);H)$. Thus the problem

$$\frac{d^2u(t)}{dt^2}+\rho(t)A^2u(t)=f(t) \qquad (23)$$

$$u(0)=0, u(T)=0 \tag{24}$$

regularly solvable. To prove the regular solvability of problem (21), (22), we will use the following inequalities

$$\left\|A^2 u\right\|^2_{L_2((0,T);H)} \le \frac{1}{\alpha}\left\|P_0 u\right\|^2_{L_2((0,T);H)} \tag{25}$$

and

$$\left\|A\frac{du}{dt}\right\|_{L_2((0,T);H)} \le \frac{1}{2\sqrt{\alpha}}\left\|P_0 u\right\|_{L_2((0,T);H)}, \tag{26}$$

which follow that from (14) and (15) with $S=0$. Next, writing problem (21), (22) in the form

$$Pu = P_0 u + Pu_1 : u(t)\in W_2^2((0,T);(0,0)),\ f(t)\in L_2((0,T);H)$$

and repeating the arguments in the proof of theorem 2, we complete the proof of theorem. Consider one example.

**Example.** Let $Q=(0,T)\times(0,\pi)$. In the area $Q$ consider the following boundary value problem

$$P\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u(x,t)=\frac{\partial^2 u(x,t)}{dt^2}+\rho(t)\frac{\partial^2 u(x,t)}{\partial x^2}+p(x,t)\frac{\partial^2 u(x,t)}{\partial x\partial t}+q(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} \tag{27}$$

$$u(x,0)=(x,T)=0,\ u(t,0)=(t,\pi)=0 \tag{28}$$

where $f(x,T)\in L_2(Q)$, $0<\alpha\le\rho(t)<\beta$, $p(x,T)$, $q(x,T)$ measurable and bounded functions in the domain $Q$. Define in space $L_2(0;\pi)$ operator $A^2 y(x)=-y''(x)$, $D(A^2)=\{y: y'$ is an absolute continuous function on $[0,\pi], y''(x)\in L_2(0,\pi)$, $A_1(t)y(x)=p_1(x,t)y'(x)\}$, $D(A_1)=\{y: y(x)$ is an absolute continuous function on $[0,\pi]$, $y'(x)\in L_2(0,\pi)$ and $A_2(t)y(x)=q_1(x,t)y(x)\}$, $D(A_2)=\{y(x): y(x)\in L_2(0,\pi)\}$.

Then it is obvious that

$$\sup_t\left\|A_1(t)A^{-1}\right\|=\sup_{(t,x)\in Q}|p(t,x)|,\quad \sup_t\left\|A_2(t)A^{-2}\right\|=\sup_{(t,x)\in Q}|q(t,x)|.$$

Then from theorem 3 we obtain the following

**Theorem 4**. *Let* $\rho(t)$ *measurable function on* $[0,T]$, *moreover* $0<\alpha\le\rho(t)<\beta<\infty$, $p(x,t)$, $q(x,t)$ *measurable bounded functions in the domain* $Q$, *moreover,*

$$\frac{1}{2\sqrt{\alpha}}\sup_{(t,x)\in Q}|p(t,x)|+\frac{1}{\alpha}\sup_{(t,x)\in Q}|q(t,x)|<1/\backslash$$

*Then for any* $f(x,t)\in L_2(Q)$ *exists a vector function* $u(x,t)\in W_{2,2}^2(Q)$, *which satisfies equation (27) in* $Q$ *almost everywhere on the border* $Q$ *condition (28) and we have the estimates*

$$\iint_Q\left(\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right|^2+\left|\frac{\partial^2 u(x,t)}{\partial t^2}\right|\right)dx\,dt\le const\iint_Q|f(t,x)|^2\,dx\,dt.$$

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